Synopsis of the Ph.D. Thesis

An Investigation of Multivariate Fractal Approximation and Fractal Operator on Various Function Spaces

Submitted by

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1. Introduction and Motivation

For many scientific and engineering problems, interpolation of a given data set and approximation of a function are indispensable. There are numerous methods for interpolation and approximation, which form important topics in classical numerical analysis and approximation theory. The nature of the function to be used for interpolation depends on the signal or image that the function is intended to model. Smoothness and non-smoothness being one of the significant features sought for the constructed interpolant, to address the interpolation of a more complicated and irregular data set, Barnsley presented the concept of fractal interpolation function (FIF) [3]. It was further taken up by many researchers; see, for instance, [4, 5, 14–16, 18]

In its basic setting, FIF is a continuous univariate function such that: (i) the function interpolates a prescribed finite data set, (ii) the graph of the function is a fractal (self-referential set) in the sense that it is a fixed point of the so-called Hutchinson-Barnsley operator corresponding to a suitable iterated function system (IFS), which is a standard framework for constructing fractals [13]. In an analytical framework, FIFs are obtained as the fixed points of the Read-Bajraktarević (RB) operators. Consequently, a FIF satisfies a self-referential equation, and the theory of FIFs has become an ideal approach for the approximation of naturally occurring functions. Differentiable FIFs supplement the classical smooth interpolation methods [9, 20]. The literature on FIFs is too vast, and therefore no effort is made here to survey this topic. Instead, in what follows, we will focus on certain facts about a few generalizations and a specific formulation of FIFs that have impacted our own work in this thesis.

Let $[a,b]$ be a closed bounded interval in $\mathbb{R}$, and $\mathbb{Y}$ be a compact arc wise connected metric space. In [22], Secelean proved that for a given countable system of data (CSD) in $[a,b] \times \mathbb{Y}$, there exists a countable IFS whose attractor is the graph of a function interpolating the given data. Henceforth, we refer to this as a countable FIF to distinguish it from the traditional FIF by Barnsley, which deals with a finite set of data points and which is based on the theory of finite IFSs.

The notion of the zipper, which is closely related to IFS, provides another methodology to create fractals [2]. As the notion of FIF (based on IFS theory) has garnered significant attention in interpolation and approximation theory, one is prompted to ask whether an interpolation scheme based on the concept of zipper can be developed. Recently, in [10], authors presented a univariate zipper fractal interpolation function for finite data sets that encompass the standard affine FIF as a special case. However, so far, there has only been a cursory treatment of the role of zipper in approximation and interpolation theories.

During the literature survey, we observed that: (1) the notion of zipper has been used for fractal interpolation resulting the so-called univariate zipper fractal interpolation for a prescribed finite data set, (2) there have been attempts toward bivariate versions of Barnsley's theory of FIF. A gap in the literature as observed after studying the above works is to find if a zipper fractal interpolation function for a countable data set can be constructed. In the present work, Chapter 2 aims to fill this gap. Similarly, the bivariate analog of FIF for countable data is hitherto unexplored. An attempt is made in this direction in the first section of Chapter 3.

A main offspring of FIF - referred to as the α-fractal function - established a close connection between univariate fractal interpolation and approximation. The notion of α-fractal function, brought to the limelight by Navascueés [19], provides a parameterized family of self-referential functions that interpolate a given function at a finite number of nodes. Further, this family of functions provides an approximation procedure in various function spaces [23]. Our specific attention to the α-fractal function is due to its potential to connect the FIF with other branches of mathematics, such as approximation theory, harmonic analysis, and functional analysis.

In parallel or even prior to the development of the univariate α-fractal functions, several works on bivariate and a few on multivariate FIFs have been reported in the literature; see, for instance, [6, 7, 11, 12, 17, 21, 24]. On one hand, these studies on the bivariate and multivariate FIFs have favored a more interpolation viewpoint. On the other hand, these constructions are not general enough to provide a multivariate analog of the α-fractal functions. We take up the study of a general framework to construct multivariate FIFs, associated α-fractal functions, and the fractal operator in various function spaces as the main focus of the thesis. This is motivated by the need to connect multivariate FIFs further with the theory of approximations and other branches of mathematics, with the α-fractal function formalism as a vehicle.

2. A Concise Description of the Research Work

The proposed thesis is divided into seven main chapters. The first two chapters (after the introductory chapter) of this thesis aim to fill some gaps observed during our literature survey on univariate and bivariate FIFs. As the title indicates, the other chapters deal with the multivariate FIFs. We would like to stress that the α-fractal function formalism of FIF and the associated fractal operator act as a recurrent theme in all the chapters. The contents of the thesis are described briefly in the following seven subsections, each of which represents a chapter in the thesis.

2.1 Introduction

The purpose of this chapter is to provide the essential background material, fix notation and terminologies, and conduct a brief literature survey relevant to our study in the subsequent chapters.

2.2 Countable Zipper Fractal Interpolation Functions

This chapter focuses on some developments in the theory of fractal interpolation of countable univariate data using the notion of zipper. The results obtained here can be seen as an extension of [10] to countable data sets and that of [22] to a more general setting, namely, zippers.

Definition 1. *Let $(\mathbb{X}, d)$ be a compact metric space and $(W_i)_{i\in\mathbb{N}}$ be a sequence of continuous maps from $\mathbb{X}$ into $\mathbb{X}$. Furthermore, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $(v_i)_{i\in\mathbb{N}_0}$ be a convergent sequence in $\mathbb{X}$ with $\lim\limits_{i\to\infty} v_i = v_\infty$, and $s := (s_i)_{i\in\mathbb{N}}$ be a binary sequence. The system $\mathcal{Z} = \{\mathbb{X}; W_i : \ i \in \mathbb{N}\}$ is called a countable zipper with vertices $(v_i)_{i\in\mathbb{N}_0}$ and signature $(s_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ if*

$$W_i(v_0) = v_{i-1+s_i}, \quad W_i(v_\infty) = v_{i-s_i} \ \ \forall \ \ i \in \mathbb{N}.$$

Note: *In what follows in this chapter and beyond, we shall use the notation $\mathbb{X}$ to denote a complete or compact metric space. The actual space $\mathbb{X}$ may change from one appearance to another.*

A nonempty closed (hence compact) set $A \subseteq \mathbb{X}$ is called an attractor of the zipper $\mathcal{Z} = \{\mathbb{X}; W_i : \ i \in \mathbb{N}\}$ if it satisfies the self-referential equation

$$A = \overline{\bigcup_{i=1}^{\infty} W_i(A)}.$$

Consider a CSD $\left\{(x_i, y_i) \in \mathbb{R}^2 : \ i \in \mathbb{N}_0\right\}$ such that the sequence of the first coordinates is strictly increasing and bounded, and the sequence of the second coordinates is convergent. Let $x_\infty = \lim_{i\to\infty} x_i, \quad y_\infty = \lim_{i\to\infty} y_i$. Set $I = [x_0, x_\infty]$. For each $i \in \mathbb{N}$, consider an affine map, $l_i : I \to [x_{i-1}, x_i] := I_i$, given by $l_i(x) = a_i x + b_i$, satisfying

$$l_i(x_0) = x_{i-1+s_i}, \quad l_i(x_\infty) \ = x_{i-s_i}. \tag{1}$$

Let us denote the Lipschitz constant of a Lipschitz continuous function $h : I \to \mathbb{R}$ by $[h]_L$. Let $J \subset \mathbb{R}$ be a sufficiently large compact interval which contains the sequence $(y_i)_{\ i\in\mathbb{N}_0}$ and y_∞. Set

$$\mathbb{X} = I \times J. \tag{2}$$

For $i \in \mathbb{N}$, let $\alpha_i : I \to \mathbb{R}$ and $q_i : I \to \mathbb{R}$ be arbitrary but fixed Lipschitz continuous functions, $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, and

$$|||\alpha|||_\infty = \sup_{i\in\mathbb{N}} \|\alpha_i\|_\infty < 1, \quad \sup_{i\in\mathbb{N}} \ [\alpha_i]_L < \infty, \quad \sup_{i\in\mathbb{N}} \ [q_i]_L < \infty. \tag{3}$$

Let $F_i : I \times J \to J$ be given by

$$F_i(x, y) = \alpha_i(x) y + q_i(x). \tag{4}$$

Set

$$W_i(x, y) = \big(l_i(x), F_i(x, y)\big) \tag{5}$$

Theorem 2.1. *Consider the countable zipper $\mathcal{Z} = \{\mathbb{X}; W_i : \ i \in \mathbb{N}\}$ with vertices $\big((x_i, y_i)\big)_{i\geq 0}$ and signature $(s_i)_{\ i\in\mathbb{N}}$ given above in (1)-(5). Further, assume that $F_i(x_0, y_0) \ = \ y_{i-1+s_i}$ and $F_i(x_\infty, y_\infty) \ = \ y_{i-s_i}$ for all $i \in \mathbb{N}$. Then there exists a unique continuous function $g_s^\alpha : I \to \mathbb{R}$ such that $g_s^\alpha(x_i) = y_i$ for all $i \in \mathbb{N}_0$, whose graph is the attractor of the countable zipper $\mathcal{Z}$.*

Definition 2. *The function g_s^α occurring in the previous theorem is referred to as the countable zipper fractal interpolation function (CZFIF).*

Remark 1. *In view of Theorem 2.1 it follows that the CZFIF g_s^α satisfies the iterative functional equation*

$$g_s^\alpha(x) \;=\; F_i\big(l_i^{-1}(x), g_s^\alpha \circ l_i^{-1}(x)\big) \quad \forall\, x \in I_i,\; i \in \mathbb{N}. \tag{6}$$

Further, $g_s^\alpha(x_\infty) = y_\infty$.

Let $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$ be a prescribed CSD. Consider a specific type of a countable zipper $\{\mathbb{X}; W_i(x,y) = (l_i(x), F_i(x,y)),\; i \in \mathbb{N}\}$ defined via

$$F_i(x,y) = \alpha_i(x)y + \phi(l_i(x)) - \alpha_i(x)b(x), \tag{7}$$

where ϕ is a Lipschitz function interpolating Δ and b is a Lipschitz function interpolating the points (x_0, y_0) and (x_∞, y_∞). Consider the perturbed data set $\hat{\Delta} := \{(x_i, \hat{y}_i) : i \in \mathbb{N}_0\}$. Assume that J is sufficiently large interval to include the ordinates of $\hat{\Delta}$ and the constant $\hat{y}_\infty = \lim_{i\to\infty} \hat{y}_i$. We shall fix the same signature as in the zipper for the original data $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$. Now we consider a countable zipper $\{\mathbb{X}; \hat{W}_i(x,y) = (l_i(x), \hat{F}_i(x,y)),\; i \in \mathbb{N}\}$ defined through the maps $l_i : I \to I_i$ as above and

$$\hat{F}_i(x,y) = \alpha_i(x)y + \hat{\phi}(l_i(x)) - \alpha_i(x)\hat{b}(x), \tag{8}$$

where $\hat{\phi}$ is a Lipschitz function interpolating $\hat{\Delta}$ and $\hat{b}$ is a Lipschitz function interpolating $(x_0, \hat{y}_0)$ and $(x_\infty, \hat{y}_\infty)$.

Theorem 2.2. *(Stability) Let $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$ and $\hat{\Delta} := \{(x_i, \hat{y}_i) : i \in \mathbb{N}_0\}$ be two CSDs. Let g_s^α be the CZFIF for Δ generated by the countable zipper $\{\mathbb{X}; W_i(x,y) = (l_i(x), F_i(x,y)),\; i \in \mathbb{N}\}$ defined via (1)-(5) and (7). Assume further that $\hat{g}_s^\alpha$ is the CZFIF corresponding to the data set $\hat{\Delta}$ generated by the zipper $\{\mathbb{X}; \hat{W}_i(x,y) = (l_i(x), \hat{F}_i(x,y)),\; i \in \mathbb{N}\}$ defined through (1)-(5) and (8). Then we have*

$$\|g_s^\alpha - \hat{g}_s^\alpha\|_\infty \le \frac{\|\phi - \hat{\phi}\|_\infty + \|\|\alpha\|\|_\infty \|b - \hat{b}\|_\infty}{1 - \|\|\alpha\|\|_\infty}.$$

Let g_s^α be the CZFIF for the data $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$ generated by the zipper $\{\mathbb{X}; W_i(x,y) = (l_i(x), F_i(x,y)),\ i \in \mathbb{N}\}$, where the maps F_i are as given in (7). Let $T_i : \mathbb{X} \to \mathbb{R},\ i \in \mathbb{N}$ be defined by

$$T_i(x,y) = [\alpha_i(x) + \varepsilon_i \psi_i(x)]y + \phi(l_i(x)) - [\alpha_i(x) + \varepsilon_i \psi_i(x)]b(x) + \xi_i \eta_i(x), \tag{9}$$

where ϕ is a Lipschitz function interpolating Δ and b is a Lipschitz function interpolating the points (x_0, y_0) and (x_∞, y_∞). Furthermore, $\varepsilon_i, \xi_i \in \mathbb{R}$ satisfy $0 < \|\|\varepsilon\|\|_\infty := \sup_{i\in\mathbb{N}} |\varepsilon_i| \le \kappa < 1$, $0 < \|\|\xi\|\|_\infty := \sup_{i\in\mathbb{N}} |\xi_i| \le \hat{\kappa} < 1$, and ψ_i, η_i are Lipschitz functions such that $\|\|\alpha\|\|_\infty + \|\|\varepsilon\|\|_\infty \|\|\psi\|\|_\infty < 1$ and $\eta_i(x_0) = \eta_i(x_\infty) = 0$.
It is easy to check that

$$T_i(x_0, y_0) = y_{i-1+s_i}, \quad T_i(x_\infty, y_\infty) = y_{i-s_i}.$$

The function T_i can be treated as a perturbation of the function F_i. Let $g_{s,\xi}^{\alpha,\varepsilon}$ be the CZFIF for the CSD $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$ corresponding to the zipper $\{\mathbb{X}; (l_i(x), T_i(x,y)), i \in \mathbb{N}\}$. The next results point to the sensitivity of the CZFIF to the perturbation in the mapping of the zipper.

Theorem 2.3. *(Sensitivity) Let g_s^α and $g_{s,\xi}^{\alpha,\varepsilon}$ be the CZFIFs for the data $\Delta = \{(x_i, y_i) : i \in \mathbb{N}_0\}$ corresponding to the zippers $\{\mathbb{X}; (l_i(x), F_i(x,y)), i \in \mathbb{N}\}$ and $\{\mathbb{X}; (l_i(x), T_i(x,y)), i \in \mathbb{N}\}$ respectively. Then*

$$\begin{aligned} \|g_{s,\xi}^{\alpha,\varepsilon} - g_s^\alpha\|_\infty \le& \frac{\|\phi - b\|_\infty \|\|\psi\|\|_\infty}{(1 - \|\|\alpha\|\|_\infty)(1 - \|\|\alpha\|\|_\infty - \|\|\varepsilon\|\|_\infty \|\|\psi\|\|_\infty)} \|\|\varepsilon\|\|_\infty \\ &+ \frac{\|\|\eta\|\|_\infty}{(1 - \|\|\alpha\|\|_\infty)(1 - \|\|\alpha\|\|_\infty - \|\|\varepsilon\|\|_\infty \|\|\psi\|\|_\infty)} \|\|\xi\|\|_\infty, \end{aligned}$$

where

$$\|\|\psi\|\|_\infty := \sup_{i\in\mathbb{N}} \{\|\psi_i\|_\infty\} < \infty \quad \textit{and} \quad \|\|\eta\|\|_\infty = \sup_{i\in\mathbb{N}} \{\|\eta_i\|_\infty\} < \infty.$$

Let $I = [x_0, x_\infty]$ be a compact interval in $\mathbb{R}$. Let us denote the space of all real-valued Lipschitz continuous functions on I by $\mathrm{Lip}(I)$. Let $f \in \mathrm{Lip}(I)$, referred to as the seed function or germ function. Suppose $\Delta = \{x_0, x_1, \ldots\}$ be an ordered set of strictly increasing points in the interval $I=[x_0,x_\infty]$ such that $\sup_{i\in\mathbb{N}_0} x_i = x_\infty$. We refer to Δ as a *partition* of I. By a slight abuse of notation, let us write $\Delta = \{(x_i, f(x_i)) : i \in \mathbb{N}_0\}$. Next, let $b: I \to \mathbb{R}$ be a fixed Lipschitz continuous function such that $b(x_0) = f(x_0)$ and $b(x_\infty) = f(x_\infty)$. Following the terminology in the literature on fractal interpolation, we call b as a *base function*. We may assume $b \neq f$ to avoid trivialities. Furthermore, we consider the following special type of maps F_i, $i \in \mathbb{N}$ defined on $\mathbb{X}$.

$$F_i(x,y) = \alpha_i(x)y + f(l_i(x)) - \alpha_i(x)b(x).$$

Consider the countable zipper $\mathcal{Z} = \{\mathbb{X}; \big(l_i(x), F_i(x,y)\big) : \ i \in \mathbb{N}\}$ and apply the countable zipper fractal interpolation method to the CSD Δ above. The interpolant obtained here is denoted by $f^{\alpha,b}_{s,\Delta}$, which satisfies the functional equation

$$f^{\alpha,b}_{s,\Delta}(x) = f(x) + \alpha_i\big(l_i^{-1}(x)\big)\Big(f^{\alpha,b}_{s,\Delta}\big(l_i^{-1}(x)\big) - b\big(l_i^{-1}(x)\big)\Big), \ \forall \ x \in I_i = [x_{i-1}, x_i].$$

Let us recall that $f^{\alpha,b}_{s,\Delta}(x_i) = f(x_i), \ \forall \ i \in \mathbb{N} \quad \text{and} \quad f^{\alpha,b}_{s,\Delta}(x_\infty) = f(x_\infty).$

Definition 3. *The aforementioned fractal function $f^{\alpha,b}_{s,\Delta}$ is called (s,α)- zipper fractal function associated to f with respect to the partition Δ and base function b.*

Let us choose b via an operator as follows. Suppose that $L : \mathrm{Lip}(I) \to \mathrm{Lip}(I)$ is an operator such that $L(f)(x_0) = f(x_0)$ and $L(f)(x_\infty) = f(x_\infty)$. Let $b = L(f)$. In this case, the corresponding (s,α)-zipper fractal function will be denoted by $f^{\alpha,L}_{s,\Delta}$ or simply by f^{α}_{s}.

Definition 4. *Let Δ, s, α and L be fixed. Associating each fixed $f \in Lip(I)$ to its fractal counterpart $f^{\alpha,L}_{s,\Delta}$, we obtain an operator called the $\mathcal{F}^{\alpha,L}_{s,\Delta}$-operator or zipper fractal operator defined as follows.*

$$\mathcal{F}^{\alpha,L}_{s,\Delta} : Lip(I) \subset \mathcal{C}(I) \to \mathcal{C}(I); \quad \mathcal{F}^{\alpha,L}_{s,\Delta}(f) = f^{\alpha,L}_{s,\Delta}.$$

Further results in this chapter aim to analyze some fundamental properties, such as closedness, relative closedness, closability, relative closability, and various types of boundedness of the nonlinear zipper fractal operator defined on $\mathrm{Lip}(I)$. Further, we shall extend this operator to $\mathcal{C}(I)$ using standard density argument.

2.3 Countable Bivariate Fractal Interpolation Functions

In this chapter, we develop a fractal interpolation technique for bivariate countable data lying on grids of a rectangle.

Consider a bivariate CSD $\Delta = \{(x_i, y_j, z_{ij}) : i, j \in \mathbb{N}_0\} \subset \mathbb{R}^3$, where (i) the sequences $(x_i)_{i\in\mathbb{N}_0}$ and $(y_j)_{j\in\mathbb{N}_0}$ are strictly increasing and bounded, (ii) the double sequence (z_{ij}) is convergent in the sense that $\lim_{i,j\to\infty} z_{ij}$ exists and it is finite, and (iii) $\lim_{j\to\infty} z_{ij} < \infty$ for each fixed $i \in \mathbb{N}_0$, and $\lim_{i\to\infty} z_{ij} < \infty$ for each fixed $j \in \mathbb{N}_0$. For a bivariate CSD, we denote $z_{\infty\infty} := \lim\limits_{i,j\to\infty} z_{ij}$. Let $x_\infty := \lim\limits_{i\to\infty} x_i$, and $y_\infty := \lim\limits_{j\to\infty} y_j$. Set $I := [x_0, x_\infty]$ and $J := [y_0, y_\infty]$. Assume that K is a sufficiently large compact interval containing the set $\{z_{ij} : i, j \in \mathbb{N}_0\} \cup \{z_{\infty\infty}\}$ and $\mathbb{X} := I \times J \times K$.

For $i \in \mathbb{N}_0$, let $s_i := \frac{1+(-1)^i}{2}$. Define $\tau : \mathbb{N} \times \{0, \infty\} \to \mathbb{N}$ by $\tau(i, 0) := i - 1 + s_i$ and $\tau(i, \infty) := i - s_i$. For $i, j \in \mathbb{N}$, let $u_i : I \to I_i := [x_{i-1}, x_i]$ and $v_j : J \to J_j := [y_{j-1}, y_j]$ be given by $u_i(x) = a_i x + b_i$ and $v_j(y) = c_j + d_j$, respectively. The constants a_i, b_i, c_j and d_j are determined by the constraints

$$
\begin{aligned}
u_i(x_0) &= s_{i-1}x_{i-1} + s_i x_i, \qquad & u_i(x_\infty) &= s_i x_{i-1} + s_{i+1} x_i,\\
v_j(y_0) &= s_{j-1}y_{j-1} + s_j y_j, \qquad & v_j(y_\infty) &= s_j y_{j-1} + s_{j+1} y_j,
\end{aligned}
$$

For every $(i, j) \in \mathbb{N} \times \mathbb{N}$ we consider the constants $\delta_i, \lambda_j, \alpha_{ij} \in (0, \infty)$. Suppose the functions $F_{ij} : \mathbb{X} \to K$ are such that the following assertions hold:

$$
\limsup_i \delta_i = \limsup_j \lambda_j = \limsup_j \sup_i \alpha_{ij} = \limsup_i \sup_j \alpha_{ij} = 0;
$$

$$
|||\alpha|||_\infty := \sup_{i,j} \alpha_{ij} < 1,
$$

and, for every $(x, y, z), (x', y', z') \in \mathbb{X}$,

$$
\left|F_{ij}(x, y, z) - F_{ij}(x', y', z)\right| \le \delta_i |x - x'| + \lambda_j |y - y'|,
$$

$$
\left|F_{ij}(x, y, z) - F_{ij}(x, y, z')\right| \le \alpha_{ij} |z - z'|,
$$

$$
F_{ij}(x_k, y_l, z_{kl}) = z_{\tau(i,k),\tau(j,l)} \quad \forall\, k, l \in \{0, \infty\}.
$$

Define $W_{ij} : \mathbb{X} \to \mathbb{X}$ by $W_{ij}(x, y, z) := \big(u_i(x), v_j(y), F_{ij}(x, y, z)\big)$. Then $\{\mathbb{X}, (W_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}\}$ is a CIFS.

Theorem 2.4. *Let us consider the CIFS $\{\mathbb{X}, W_{ij} : (i,j) \in \mathbb{N}\times\mathbb{N}\}$ defined above. Assume that for each $(i,j) \in \mathbb{N}\times\mathbb{N}$, the function $F_{ij} : \mathbb{X} \to K$ further satisfies the following matching conditions*

1. *for all $i \in \mathbb{N}$ and $x^* = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)$, $F_{ij}(x^*, y, z) = F_{i+1,j}(x^*, y, z)$, $\forall\, y \in J, z \in K$,*
2. *for all $j \in \mathbb{N}$ and $y^* = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)$, $F_{ij}(x, y^*, z) = F_{i+1,j}(x, y_\infty, z)$, $\forall\, x \in I, z \in K$.*

Then there exists a unique continuous function $g : I \times J \to \mathbb{R}$ such that $g(x_i, y_j) = z_{ij}$ for all $i, j \in \mathbb{N}_0 \times \mathbb{N}_0$, and the graph of g is the attractor of the CIFS defined above.

In order to explore some approximation theoretic aspects, we consider here a special case of the countable bivariate FIF constructed previously. To this end, let $I\times J = [x_0, x_\infty] \times [y_0, y_\infty] \subset \mathbb{R}^2$. We say that $\Delta = \{x_i : i \in \mathbb{N}_0\} \times \{y_j : j \in \mathbb{N}_0\} \subset I \times J$ is a partition of $I \times J$ if the sequences $(x_i)_{i\in\mathbb{N}_0}$ and $(y_j)_{i\in\mathbb{N}_0}$ are strictly increasing such that $\lim\limits_{i\to\infty} x_i = x_\infty$ and $\lim\limits_{j\to\infty} y_j = y_\infty$.

Let $\mathrm{Lip}(I\times J) \subset \mathcal{C}(I \times J)$ denote the set of all Lipschitz continuous real-valued functions defined on $I \times J$. Fix $f \in \mathrm{Lip}(I\times J)$ and by a slight abuse of notation, consider the bivariate CSD $\Delta = \big\{(x_i, y_j, f(x_i, y_j)) : i, j \in \mathbb{N}_0\big\}$. Assume that $L : \mathrm{Lip}(I\times J) \to \mathrm{Lip}(I\times J)$ is an operator satisfying the boundary conditions $L(f)(x_k, y_l) = f(x_k, y_l) \quad$ for all $\quad k, l \in \{0, \infty\}$. Let K be a sufficiently large compact interval containing the set $\{f(x_i, y_j) : i, j \in \mathbb{N}_0\}$ and $\mathbb{X} = I \times J \times K$. For $i, j \in \mathbb{N}$, define $F_{ij} : \mathbb{X} \to K$ by

$$F_{ij}(x,y,z) := \alpha(u_i(x), v_j(y))z + f(u_i(x), v_j(y)) - \alpha(u_i(x), v_j(y))L(f)(x,y),$$

where $\alpha : I \times J \to \mathbb{R}$ is a Lipschitz continuous function such $\alpha_{ij} = \|\alpha\|_{\infty, I_i\times J_j} := \sup_{(x,y)\in I_i\times J_j} \big|\alpha(x,y)\big|$ satisfy the conditions required in the above theorem.

Theorem 2.5. *Assume that the partition Δ, scaling function α, and operator L are fixed. Then corresponding to each $f \in Lip(I\times J)$, there exists a unique continuous function $f^\alpha_{\Delta,L} : I \times J \to \mathbb{R}$ such that*

1. *$f^{\alpha}_{\Delta,L}$ interpolates f at the points in Δ, that is, $f^{\alpha}_{\Delta,L}(x_i,y_j) = f(x_i,y_j)$ for all $(x_i,y_j) \in \Delta$,*

2. *the graph of $f^{\alpha}_{\Delta,L}$ is the attractor of the CIFS $\{\mathbb{X}, W_{ij} : (i,j) \in \mathbb{N}\times\mathbb{N}\}$ defined above.*

Definition 5. *The function $f^{\alpha}_{\Delta,L}$ is referred to as the (countable bivariate) α-fractal function associated to the germ function f, with respect to the parameters α, Δ and L.*

Definition 6. *Let α, Δ and L be fixed. The operator $\mathcal{F}^{\alpha}_{\Delta,L} : Lip(I\times J) \subset \mathcal{C}(I\times J) \to \mathcal{C}(I\times J)$, defined by*

$$\mathcal{F}^{\alpha}_{\Delta,L}(f) := f^{\alpha}_{\Delta,L},$$

which assigns to each $f \in Lip(I\times J)$ its self-referential counterpart $f^{\alpha}_{\Delta,L}$, is called the α-fractal operator on Lip($I\times J$).

The subsequent parts of this chapter focus on studying the approximation and operator theoretic properties of the bivariate fractal operator $\mathcal{F}^{\alpha}_{\Delta,L}$.

2.4 Multivariate Fractal Interpolation Function on Rectangular Grids

In this chapter, we demonstrate a general framework for the construction of multivariate FIF that is amenable to the α-fractal function formalism, as mentioned in the introductory section.

Let $n \geq 2$, $\Delta = \left\{(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}, y_{i_1 i_2 \dots i_n}) : i_k \in \Sigma_{N_k,0}; k \in \Sigma_n\right\}$ is a data set such that $x_{k,0} < x_{k,1} < \cdots < x_{k,N_k}$ for each $k \in \Sigma_n$; $n \geq 2$. For $k = 1,2,\dots n$, set $I_k = [x_{k,0}, x_{k,N_k}]$ and $\Omega = \prod_{k=1}^{n} I_k$. To simplify the notation, for $m \in \mathbb{N}$, we write $\Sigma_m = \{1,2,\dots,m\}$, $\Sigma_{m,0} = \{0,1,\dots m\}$, $\partial\Sigma_{m,0} = \{0,m\}$, and $\mathrm{int}\Sigma_{m,0} = \{1,2,\dots,m-1\}$. Furher, we shall denote by I_{k,i_k}, the typical subinterval of I_k determined by the partition $\{x_{k,0}, x_{k,1}, \dots, x_{k,N_k}\}$, $I_{k,i_k} = [x_{k,i_k-1}, x_{k,i_k}]$ for $i_k \in \Sigma_{N_k}$. For any $i_k \in \Sigma_{N_k}$, let $u_{k,i_k} : I_k \to I_{k,i_k}$ be an affine map satisfying

$$\begin{cases} u_{k,i_k}(x_{k,0}) = x_{k,i_k-1} \text{ and } u_{k,i_k}(x_{k,N_k}) = x_{k,i_k}, & \text{if } i_k \text{ is odd,} \\ u_{k,i_k}(x_{k,0}) = x_{k,i_k} \text{ and } u_{k,i_k}(x_{k,N_k}) = x_{k,i_k-1}, & \text{if } i_k \text{ is even,} \end{cases}$$

$$\left|u_{k,i_k}(x)-u_{k,i_k}(x')\right| \le \alpha_{k,i_k}|x-x'|, \quad \forall\ x,x' \in I_k,$$

where $0 \le \alpha_{k,i_k} < 1$ is a constant. Let $\tau : \mathbb{Z}\times\{0,N_1,N_2,\ldots,N_n\} \to \mathbb{Z}$ be defined by

$$\begin{cases} \tau(i,0)=i-1 \ \text{ and } \ \tau(i,N_k)=i, \ \text{ if } \ i \ \text{ is odd,} \\ \tau(i,0)=i, \ \text{ and } \ \tau(i,N_k)=i-1, \quad \text{if } \ i \ \text{ is even.} \end{cases}$$

Let $\mathbb{X} := \Omega\times\mathbb{R}$. For each $(i_1,i_2,\ldots,i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}$, let $F_{i_1 i_2 \ldots i_n} : \mathbb{X} \to \mathbb{R}$ be a continuous function satisfying the following conditions.

$$F_{i_1 i_2 \ldots i_n}(x_{1,j_1},x_{2,j_2},\ldots,x_{n,j_n},y_{j_1 j_2 \ldots j_n}) = y_{\tau(i_1,j_1)\tau(i_2,j_2)\ldots\tau(i_n,j_n)},$$

for all $(j_1,j_2,\ldots,j_n) \in \prod_{k=1}^{n}\partial\Sigma_{N_k,0}$ and

$$\left|F_{i_1 i_2 \ldots i_n}(x_1,x_2,\ldots,x_n,y)-F_{i_1 i_2 \ldots i_n}(x_1,x_2,\ldots,x_n,y')\right| \le \gamma_{i_1 i_2 \ldots i_n}|y-y'|,$$

for all $(x_1,x_2,\ldots,x_n) \in \Omega$ and $y,y' \in \mathbb{R}$, where $0 \le \gamma_{i_1 i_2 \ldots i_n} < 1$ is a constant. Finally, for each $(i_1,i_2,\ldots,i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}$, we define $W_{i_1 i_2 \ldots i_n} : K \to K$ by

$$\begin{aligned} W_{i_1 i_2 \ldots i_n}(x_1,x_2,\ldots,x_n,y) = \big(&u_{1,i_1}(x_1),u_{2,i_2}(x_2),\ldots,u_{n,i_n}(x_n), \\ &F_{i_1 i_2 \ldots i_n}(x_1,x_2,\ldots,x_n,y)\big). \end{aligned}$$

and consider the Iterated Function System (IFS)

$$\Big\{\mathbb{X}, W_{i_1 i_2 \ldots i_n} : (i_1,i_2,\ldots i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}\Big\}.$$

Theorem 2.6. *Let $\Delta = \big\{(x_{1,i_1},x_{2,i_2},\ldots,x_{n,i_n},y_{i_1 i_2 \ldots i_n}) : i_k = 0,1,\ldots,N_k; k \in \Sigma_n\big\}$ be a prescribed multivariate data set and $\{K, W_{i_1 i_2 \ldots i_n} : (i_1,i_2,\ldots,i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}\}$ be the IFS associated to it, as defined above. Assume that for each $(i_1,i_2,\ldots,i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}$, the map $F_{i_1 i_2 \ldots i_n}$ satisfy the following matching conditions:*
For all $i_k \in int\Sigma_{N_k,0}$, $1 \le k \le n$, $(i_1,i_2,\ldots,i_n) \in \prod_{k=1}^{n}\Sigma_{N_k}$ and $x_k^* = u_{k,i_k}^{-1}(x_{k,i_k}) = u_{k,i_k+1}^{-1}(x_{k,i_k})$,

$$\begin{aligned} &F_{i_1 \ldots i_k \ldots i_n}(x_1,\ldots,x_{k-1},x_k^*,x_{k+1},\ldots x_n,y) \\ &= F_{i_1 \ldots i_k+1 \ldots i_n}(x_1,\ldots,x_{k-1},x_k^*,x_{k+1}\ldots x_n,y), \end{aligned}$$

where $(x_1, \dots x_{k-1}, x_{k+1} \dots, x_n) \in \prod_{j=1, j\neq k}^{n} I_j$ *and* $y \in \mathbb{R}$*. Then there exists a unique continuous function* $\tilde{f} : \Omega \to \mathbb{R}$ *interpolating* Δ *whose graph is the attractor of the IFS considered above.*

Now, we obtain a parameterized family of fractal functions associated with a prescribed germ function f by using the idea of multivariate fractal interpolation demonstrated above. To this end, consider the set

$$\Delta = \Big\{(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \in \Omega \subset \mathbb{R}^n : i_k \in \Sigma_{N_k,0},\ k \in \Sigma_n\Big\},$$

where $x_{k,0} < x_{1,k} < \cdots < x_{k,N_k}$ for each $k \in \Sigma_n := \{1, 2, \dots, n\}$. With a slight abuse of notation, let us write

$$\Delta = \Big\{\big(x_{1,i_1}, \dots, x_{n,i_n}, f(x_{1,i_1}, \dots, x_{n,i_n})\big) \in \Omega \times \mathbb{R} : i_k \in \Sigma_{N_k,0},\ k \in \Sigma_n\Big\}.$$

Choose a function $b \in \mathcal{C}(\Omega)$ such that for all $(j_1, j_2, \dots, j_n) \in \prod_{k=1}^{n} \partial\Sigma_{N_k,0}$,

$$b(x_{1,j_1}, x_{2,j_2}, \dots, x_{n,j_n}) = f(x_{1,j_1}, x_{2,j_2}, \dots, x_{n,j_n}).$$

Consider a continuous map $\alpha : \Omega \to \mathbb{R}$ such that $\|\alpha\|_\infty < 1$. Define

$$\begin{aligned} &F_{i_1 i_2 \dots i_n}\big(x_1, x_2, \dots, x_n, y\big) \\ &= f\big(u_{1,i_1}(x_1), u_{2,i_2}(x_2) \dots u_{n,i_n}(x_n)\big) \\ &\quad + \alpha\big(u_{1,i_1}(x_1), u_{2,i_2}(x_2) \dots u_{n,i_n}(x_n)\big)\big(y - b(x_1, x_2, \dots, x_n)\big). \end{aligned}$$

The above choice of functions $F_{i_1 i_2 \dots i_n}$ in satisfy the constraints required in the previous theorem. Hence, there exists a unique fractal interpolation function, which we shall denote by $f_{\Delta,b}^{\alpha} : \prod_{k=1}^{n} I_k \to \mathbb{R}$, such that it satisfies the self-referential functional equation

$$f_{\Delta,b}^{\alpha}(X) = f(X) + \alpha(X)\Big((f_{\Delta,b}^{\alpha} - b)\big(u_{i_1 i_2 \dots i_n}^{-1}(X)\big)\Big),$$

for all $X \in \prod_{k=1}^{n} I_{k,i_k}$ and $(i_1, i_2, \dots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$, where $X = (x_1, x_2, \dots, x_n)$ and $u_{i_1 i_2 \dots i_n}^{-1}(X) = \big(u_{1,i_1}^{-1}(x_1), u_{2,i_2}^{-1}(x_2), \dots u_{n,i_n}^{-1}(x_n)\big)$. We call $f_{\Delta,b}^{\alpha} : \prod_{k=1}^{n} I_k \to \mathbb{R}$, the multivariate α-fractal interpolation function corresponding to the seed function f.

We study the fractal dimension of the graph of $f^{\alpha}_{\Delta,b}$ and its fractional integral. Next, we choose the base function $b : \Omega \to \mathbb{R}$ in the above construction via a nonlinear operator $L : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ defined by $b = L(f)$. This, as in the earlier chapters, gives rise to a fractal operator on $\mathcal{C}(\Omega)$, which is the further topic of study in this chapter.

2.5 Smoothness Preserving Multivariate Fractal Interpolation Functions

In this chapter, we investigate multivariate FIFs in $\mathcal{C}^M(\Omega)$, the function space consisting of functions whose all partial derivatives upto order M exist and are continuous. To this end, let $f \in \mathcal{C}^M(\Omega)$ be fixed. Define $\mathcal{C}^M_f(\Omega)$ by

$$\mathcal{C}^M_f(\Omega) = \big\{g \in \mathcal{C}^M(\Omega) : D^l(g)(X) = D^l(f)(X) \ \ \forall \ \ l \text{ with } |l| \le M \\ \text{and } \ X \in \partial\Omega\big\}.$$

We shall consider the space $\mathcal{C}^M(\Omega)$ equipped with the norm $\|\cdot\|_{M,\infty}$, defined by $\|g\|_{M,\infty} = \sum\limits_{|l|\le M} \|g\|_\infty$. It is plain to see that the set $\mathcal{C}^M_f(\Omega)$ endowed with the metric induced by the norm $\|\cdot\|_{M,\infty}$ is a complete metric space.

Now choose the scaling vector α and base function b appearing in the construction of the multivariate α-fractal functions $f^{\alpha}_{\Delta,b}$ corresponding to $f \in \mathcal{C}^M(\Omega)$ also as smooth enough, that is,

1. $\alpha_{i_1\ldots i_n} \in \mathcal{C}^M(\Omega)$ for $(i_1, \ldots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$.
2. $b \in \mathcal{C}^M_f(\Omega)$.

Let us denote

$$|||\alpha|||_{M,\infty} = \max\Big\{\big\|D^l(\alpha_{i_1\ldots i_n})\big\|_\infty : |l| \le M, (i_1 \ldots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}\Big\}.$$

Similar to that in the construction of $f^{\alpha}_{\Delta,b}$ given in the previous chapter, let us define an RB type operator T_f on $\mathcal{C}^M_f(\Omega)$ by

$$T_f(g)(X) = f(X) + \alpha_{i_1\ldots i_n}(X)(g-b)\big(u^{-1}_{i_1\ldots i_n}(X)\big),$$

for all $X \in \prod_{k=1}^{n} I_{i_k,k}$ and $(i_1 \dots i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$. Recall the notation

$$|a| = \min\left\{|a_{k,i_k}| : i_k \in \Sigma_{N_k}, k \in \Sigma_n\right\}.$$

Theorem 2.7. *The map T_f maps $\mathcal{C}_f^M(\Omega)$ into $\mathcal{C}_f^M(\Omega)$ and satisfies the Lipschitz condition*

$$\left\|T_f(g) - T_f(h)\right\|_{M,\infty} \leq \left(\frac{2}{|a|}\right)^{nM} \|\|\alpha\|\|_{M,\infty} \|g-h\|_{M,\infty},$$

for all $g,h \in \mathcal{C}_f^M(\Omega)$. In particular, if

$$\left(\frac{2}{|a|}\right)^{nM} \|\|\alpha\|\|_{M,\infty} < 1,$$

then T_f is a contraction map. Its unique fixed point $f_{\Delta,b}^{\alpha} \in \mathcal{C}_f^M(\Omega)$ is such that

$$D^l(f_{\Delta,b}^{\alpha})(X) = D^l(f)(X) + \sum_{p=0}^{l} \binom{l}{p} D^{l-p}\big(\alpha_{i_1\dots i_n}(X)\big) D^p\big(f_{\Delta,b}^{\alpha} - b\big)\big(u_{i_1\dots i_n}^{-1}(X)\big), \tag{10}$$

for all $X \in \prod_{k=1}^{n} I_{k,i_k}$, $(i_1,\dots,i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$ and multi-index l with $|l| \leq M$.

In the subsequent parts of the chapter, we shall use the above construction to get a fractal Hermite interpolant. To this end, we shall first extend the classical bivariate Hermite interpolation formula presented in [1] to higher dimensions. Further, we shall discuss some shape preserving approximation aspects of multivariate smooth FIF.

2.6 Multivariate Fractal Functions in Lebesgue and Sobolev Spaces

In the same spirit as in the previous chapters, we shall construct multivariate α-fractal functions corresponding to a fixed function, but this time in the function spaces: (1) Lebesgue space $\mathcal{L}^P(\Omega)$ and (2) Sobolev space $\mathcal{W}^{M,P}(\Omega)$.

Let $n \geq 2$ be an integer and $\Delta = \{(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \in \mathbb{R}^n : i_k \in \Sigma_{N_k,0}; k \in \Sigma_n\}$ be such that $x_{k,0} < x_{k,1} < \dots < x_{k,N_k}$ for each $k \in \Sigma_n$. Note that $x_{k,0} < x_{k,1} <$

$\cdots < x_{k,N_k}$ determines a partition of I_k into subintervals $I_{k,i_k} = \left[x_{k,i_{k-1}}, x_{k,i_k}\right)$ for $i_k = \in \mathrm{int}\Sigma_{N_k,0}$ and $I_{k,N_k} = \left[x_{k,N_{k-1}}, x_{k,N_k}\right]$.

It is worth to note that $I_k = \bigcup_{i_k=1}^{N_k} I_{k,i_k}$ for $k \in \Sigma_n$, and each node point in the partition of I_k is exactly in one of the subintervals I_{k,i_k}, $i_k = 1, 2 \ldots, N_k$ mentioned above.

For each $i_k \in \Sigma_{N_k}$, let $u_{k,i_k} : I_k \to I_{k,i_k}$ be an affine map of the form

$$u_{k,i_k}(x) = a_{k,i_k}x + b_{k,i_k},$$

satisfying

$$\begin{cases} u_{k,i_k}(x_{k,0}) = x_{k,i_k-1} \quad \text{and} \quad u_{k,i_k}(x_{k,N_k}) = x_{k,i_k} \quad \text{if} \quad i_k \text{ is odd}, \\ u_{k,i_k}(x_{k,0}) = x_{k,i_k} \quad \text{and} \quad u_{k,i_k}(x_{k,N_k}) = x_{k,i_k-1} \quad \text{if} \quad i_k \text{ is even}. \end{cases}$$

When the interval I_{k,i_k} involved in the definition of affine maps is half-open, the above equation needs to be interpreted in terms of the one-sided limit. For instance, when $i_k \in \mathrm{int}\Sigma_{N_k,0}$ is odd, $u_{k,i_k}(x_{k,N_k}) = x_{k,i_k}$ actually means $\lim_{x \to x_{k,N_k}^-} u_{k,i_k}(x) = x_{k,i_k}$.

Note that

$$|u_{k,i_k}(x) - u_{k,i_k}(x')| \leq \gamma_{k,i_k}|x - x'|, \quad \forall \ x, x' \in I_k,$$

for $0 \leq \gamma_{k,i_k} = |a_{k,i_k}| < 1$. Using the definition of the map u_{k,i_k}, one can verify that

$$u_{k,i_k}^{-1}(x_{k,i_k}) = u_{k,i_k+1}^{-1}(x_{k,i_k}),$$

for all $i_k \in \mathrm{int}\Sigma_{N_k,0}$.

Finally, for each $g \in \mathcal{L}^P(\Omega)$, and $X = (x_1, \ldots, x_n) \in \prod_{k=1}^n I_{k,i_k}$, $(i_1, \ldots, i_n) \in \prod_{k=1}^n \Sigma_{N_k}$, we define $T_f(g)$ as

$$T_f(g)(X) = f(X) + \alpha_{i_1 \ldots i_n}(g - b)\left(u_{i_1 \ldots i_n}^{-1}(X)\right),$$

where $u_{i_1 \ldots i_n}^{-1}(X) = \left(u_{1,i_1}^{-1}(x_1), \ldots, u_{1,i_1}^{-1}(x_n)\right)$, $b(\neq f) \in \mathcal{L}^P(\Omega)$ be arbitrary but fixed, and $\alpha_{i_1 \ldots i_n}$ are real numbers that satisfy certain constraints which will be mentioned in the sequel. The $\left(\prod_{k=1}^{n} N_k\right)$-tuple comprised of the real numbers $\alpha_{i_1 \ldots i_n}$ is called the *scaling vector* or *scaling factor* and it is denoted by α. We define

$$|||\alpha|||_\infty = \max\left\{|\alpha_{i_1\ldots i_n}| : \ (i_1, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}\right\}.$$

The main objective in this section is to choose the scale vector α and base function b so that the Read-Bajraktarević (RB) operator T_f is a well-defined map, and, in fact, T_f is a contraction map on the function $\mathcal{L}^P(\Omega)$ under suitable constraints.

Theorem 2.8. *Let $f \in \mathcal{L}^P(\Omega)$ for $1 \le P \le \infty$. Then T_f maps $\mathcal{L}^P(\Omega)$ to $\mathcal{L}^P(\Omega)$. Further, T_f is a contraction map, if*

$$\begin{cases} \left[\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} (\prod_{k=1}^{n} |a_{k,i_k}|) |\alpha_{i_1\ldots i_n}|^P\right]^{\frac{1}{P}} < 1, & \text{for} \quad 1 \le P < \infty. \\ |||\alpha|||_\infty < 1, & \text{for} \quad P = \infty. \end{cases}$$

Hence there exists a unique $f_{\Delta,b}^{\alpha} \in \mathcal{L}^P(\Omega)$ such that

$$f_{\Delta,b}^{\alpha}(X) = f(X) + \alpha_{i_1\ldots i_n}(f_{\Delta,b}^{\alpha} - b)\big(u_{i_1\ldots i_n}^{-1}(X)\big),$$

for $X \in \prod_{k=1}^{n} I_{k,i_k}$, and $(i_1, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}$.

Similarly we have the following.

Theorem 2.9. *Let $f \in \mathcal{W}^{M,P}(\Omega)$ for $1 \le P \le \infty$. Suppose that the base function $b \in \mathcal{W}^{M,P}(\Omega)$ and the scaling vector is chosen so that*

$$\begin{cases} \left[\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} \frac{|\alpha_{i_1\ldots i_n}|^P}{\prod_{k=1}^{n} |a_{k,i_k}|^{MP-1}}\right]^{\frac{1}{P}} < 1, & \text{for} \quad 1 \le P < \infty. \\ \max\{\frac{|\alpha_{i_1\ldots i_n}|}{\prod_{k=1}^{n} |a_{k,i_k}|^M} : \quad (i_1, \ldots, i_n) \in \prod_{k=1}^{n} \Sigma_{N_k}\} < 1, & \text{for} \quad P = \infty. \end{cases} \tag{11}$$

Then the RB operator T_f given in (2.6) is a contraction map on $\mathcal{W}^{M,P}(\Omega)$. Consequently, T_f has a unique fixed point $f_{\Delta,b}^{\alpha}$.

2.7 Fractal Functions in Mixed Norm Spaces

Thus far, our discussion on multivariate fractal functions has been limited to function spaces essentially endowed with the classical $\mathcal{L}^P$– norm. In this chapter, we continue this investigation to some mixed norm spaces. We shall also study the approximation properties of the multivariate Kantorovich operators and their consequences in the fractal approximation process on mixed Lebesgue spaces.

Let $n \geq 2$ be an integer and $\Omega = [0,1]^n$. Consider a partition $\Delta = \{(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \in \mathbb{R}^n : i_k \in \Sigma_{N_k,0}; k \in \Sigma_n\}$ of Ω such that $0 = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = 1$ for each $k \in \Sigma_n$. For each $k \in \Sigma_n$ $0 = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = 1$ determines a partition of $[0,1]$ into subintervals $I_{k,i_k} := [x_{k,i_{k-1}}, x_{k,i_k})$ for $i_k = 1,2,\dots,N_k - 1$ and $I_{k,N_k} := [x_{k,N_k-1}, x_{k,N_k}]$. For a fixed $f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ and an arbitrary $g \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$, define $T_f : [0,1]^n \to \mathbb{R}$ by

$$T_f(g)(X) = f(X) + \alpha_{i_1\dots i_n}(g-b)(u^{-1}_{i_1\dots i_n}(X)) \quad X \in \prod_{k=1}^{n} I_{k,i_k},$$

where $b \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ is fixed and $\alpha_{i_1\dots i_n} \in \mathbb{R}$ is chosen so as to satisfy some bound to be mentioned in the sequel. Let $\alpha = (\alpha_{i_1\dots i_n})$ be a $(\prod_{k=1}^n N_k)$-tuple of real numbers.

Theorem 2.10. *Let $\overrightarrow{P} \in [1,\infty)^n$ and $f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ and the parameters α, Δ, and $b \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ be fixed. Further, let us assume that the scale vector is chosen such that*

$$\begin{aligned} C^{\overrightarrow{P}}_{\Delta,\alpha} := \Bigg[\sum_{i_n=1}^{N_n} |a_{n,i_n}| \Bigg\{ \sum_{i_{n-1}=1}^{N_{n-1}} |a_{n-1,i_{n-1}}| \dots \\ \Bigg\{ \sum_{i_2=1}^{N_2} |a_{2,i_2}| \Bigg\{ \sum_{i_1=1}^{N_1} |\alpha_{i_1\dots i_n}|^{P_1} |a_{1,i_1}| \Bigg\}^{\frac{P_2}{P_1}} \Bigg\}^{\frac{P_3}{P_2}} \dots \Bigg\}^{\frac{P_n}{P_{n-1}}} \Bigg]^{\frac{1}{P_n}} < 1. \end{aligned}$$

Then $T_f : \mathcal{L}^{\overrightarrow{P}}([0,1]^n) \to \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ is a contraction map. Hence by Banach fixed point theorem there exists a unique $f^\alpha_{\Delta,b} \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ which satisfies the following self-referential equation

$$f^\alpha_{\Delta,b}(X) = f(X) + \alpha_{i_1\dots i_n}(f^\alpha_{\Delta,b} - b)(u^{-1}_{i_1\dots i_n}(X)), \quad X \in \prod_{k=1}^{n} I_{k,i_k}.$$

Now, let $f : [0,1]^n \to \mathbb{R}$ be a given function and $M_1, \dots, M_n$ be non-negative integers. For the sake of brevity, let us write $\mathbf{M} = (M_1, \dots, M_n)$, the Bernstein-Kantorovich polynomial associated with f is defined as

$$\begin{aligned} K_{\mathbf{M}}(f)(X) &= \sum_{i_1=0}^{M_1} \cdots \sum_{i_n=0}^{M_n} \prod_{k=1}^{n} P^k_{M_k,i_k}(x_k) \\ &\quad \int_0^1 \cdots \int_0^1 f\left(\frac{i_1+t_1}{M_1+1}, \ldots, \frac{i_n+t_n}{M_n+1}\right) \mathrm{d}t_1 \ldots \mathrm{d}t_n \end{aligned} \tag{12}$$

Lemma 2.1. *Let $K_{\boldsymbol{M}}$ be the Kantorovich operator defined in (12) and $f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$. Then $K_{\boldsymbol{M}}(f) \in \mathcal{C}([0,1]^n) \subset \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$.*

Now, for $k = 1, 2 \ldots, n$, consider $\eta_k : [0,1]^n \to [0,1] \subset \mathbb{R}$ and $\xi_k : [0,1]^n \to [0,1] \subset \mathbb{R}$ are defined by $\xi_k(X) = x_k$ and $\eta_k(X) = x_k^2$

Lemma 2.2. *Let $K_{\boldsymbol{M}}$ be the multivariate Kantorovich operator defined in (12). Then we have the following*

(1) $K_{\boldsymbol{M}}(\xi_j) = \frac{M_j}{M_j+1}\xi_j + \frac{1}{2(M_j+1)}$.

(2) $K_{\boldsymbol{M}}(\eta_j) = \frac{1}{(M_j+1)^2}\left[M_j\left(M_j-1\right)\eta_j + 2M_j\xi_j + \frac{1}{3}\right]$

As a direct consequence of the above lemma, we have the following corollary.

Corollary 2.1. *Let $K_{\boldsymbol{M}}$ be the Kantorovich operator defined in (12). Then we have the following*

(1) $\|K_{\boldsymbol{M}}(\xi_j) - \xi_j\|_\infty \to 0$ *as* $M_j \to \infty$.

(2) $\|K_{\boldsymbol{M}}(\eta_j) - \eta_j\|_\infty \to 0$ *as* $M_j \to \infty$.

(3) $\|K_{\boldsymbol{M}}(\eta) - \eta\|_\infty \to 0$ *as* $\boldsymbol{M} \to \infty$*, where the function,* $\eta : [0,1]^n \to \mathbb{R}$ *is defined by* $\eta(X) = \|X\|^2$.

In the following theorem, we show that the multiparameter sequence $(K_{\mathbf{M}}(f))$ converges to f uniformly for all $f \in \mathcal{C}([0,1]^n)$ as $\mathbf{M} \to \infty$.

Theorem 2.11. *Let $f \in \mathcal{C}([0,1]^n)$, then $\|K_{\boldsymbol{M}}(f) - f\|_\infty = 0$ as $\boldsymbol{M} \to \infty$*

Lemma 2.1 ensures that the Bernstein-Kantorovich operator maps the mixed Lebesgue space into itself. In the following lemma we prove that $K_{\mathbf{M}} : \mathcal{L}^{\overrightarrow{P}}([0,1]^n) \to \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ is in fact a bounded operator.

Lemma 2.3. *Let $K_{\boldsymbol{M}} : \mathcal{L}^{\overrightarrow{P}}([0,1]^n) \to \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ be the Bernstein-Kantorovich operator as defined above. Then*

$$\|K_{\boldsymbol{M}}(f)\|_{\overrightarrow{P}} \leq \|f\|_{\overrightarrow{P}} \quad \forall \quad f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n).$$

As in the classical Lebesgue spaces, $\mathcal{C}([0,1]^n)$ is dense in the mixed norm Lebesgue space $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$. We prove this in the following theorem.

Theorem 2.12. *Let $\overrightarrow{P} \in [1,\infty)^n$. Then $\mathcal{C}([0,1]^n)$ is dense in $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$.*

Theorem 2.13. *Let $f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ and $K_{\boldsymbol{M}}$ be the operator defined as above. Then $K_{\boldsymbol{M}}(f)$ converges to f in $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$.*

Using the multiparameter sequence of Bernstein-Kantorovich operators $(K_{\mathbf{M}})_{\mathbf{M}\in\mathbb{N}^d}$ constructed above, we shall obtain a fractal approximation process on the mixed Lebesgue space $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ for $\overrightarrow{P} \in [1,\infty)^n$. To this end, we select the base function $b : [0,1]^n \to \mathbb{R}$ involved in the construction of α-fractal functions on $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ via the Bernstein-Kantorovich operator $K_{\mathbf{M}} : \mathcal{L}^{\overrightarrow{P}}([0,1]^n) \to \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$. To be precise, we take $b = K_{\mathbf{M}}(f)$ and the corresponding α-fractal function is denoted by $f^{\alpha}_{\Delta,K_{\mathbf{M}}}$. This gives rise to an operator $\mathcal{F}^{\alpha}_{\Delta,K_{\mathbf{M}}} : \mathcal{L}^{\overrightarrow{P}}([0,1]^n) \to \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$, defined by $f \mapsto f^{\alpha}_{\Delta,K_{\mathbf{M}}}$, which we call as the Bernstein-Kantorovich Fractal operator.

Theorem 2.14. *Let the parameters α and Δ be fixed. Suppose $f \in \mathcal{L}^{\overrightarrow{P}}([0,1]^n)$ be arbitrary, then the fractal perturbation $\mathcal{F}^{\alpha}_{\Delta,K_{\boldsymbol{M}}}(f)$ of f converges to f as $\boldsymbol{M} \to \infty$.*

As a straightforward application of the fractal operator on mixed Lebesgue spaces, we shall construct a Schauder basis consisting of fractal functions for the mixed Lebesgue spaces $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$.

The set of dyadic intervals in $[0,1]$ is defined by

$$\mathcal{D} = \left\{ \left[\frac{j-1}{2^m}, \frac{j}{2^m}\right) : 1 \leq j \leq 2^m, m \geq 0 \right\}.$$

For any $I \in \mathcal{D}$, let I_0 and I_1 denote the left and right halves of I, respectively. The $\mathcal{L}^{\infty}$ - normalized Haar function h_I is defined as $h_I = \chi_{I_0} - \chi_{I_1}$. The sequence

$(h_I)_{I\in\mathcal{D}}$ is known as $\mathcal{L}^\infty$ - normalized Haar system. Let us define the collection of dyadic hyper-rectangles in $[0,1]^n$ by

$$\mathcal{R}_n = \Big\{\mathcal{I} := I_1 \times I_2 \times \ldots \times I_n : I_1, I_2, \ldots, I_n \in \mathcal{D}\Big\}.$$

The multiparameter Haar system $\{h_{\mathcal{I}} : \mathcal{I} \in \mathcal{R}_n\}$ is given by

$$h_{\mathcal{I}}(X) = \bigotimes_{k=1}^{n} h_{I_k}(x_k), \quad X \in [0,1]^n..$$

In [8, Proposition I.1], it is proved that the biparameter Haar system (that is, the multiparameter Haar system with $n = 2$) is an unconditional Schauder basis for mixed Lebesgue space $\mathcal{L}^{(P_1,P_2)}([0,1]^2)$, $1 < P_1, P_2 < \infty$. A similar computation proves that the multiparameter Haar system $\{h_{\mathcal{I}} : \mathcal{I} \in \mathcal{R}_n\}$ is an unconditional Schauder basis for mixed Lebesgue space $\mathcal{L}^{\overrightarrow{P}}([0,1]^n)$, $\overrightarrow{P} \in (1,\infty)^n$.

3. Proposed Contents of the Thesis

- **Chapter 1.** Introduction
  1.1 Fractals: An Overview
  1.2 Fractal Dimensions
  1.3 Iterated Function System
  1.4 Interpolation and Approximation: A Broad Perspective
  1.5 Univariate Fractal Interpolation
  1.6 Univariate α-Fractal Functions and Their Approximation Properties
  1.7 Some Elements of Function Spaces and Operator Theory
  1.8 Riemann-Liouville Fractional Integral
  1.9 Motivation for the Current Work
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- **Chapter 2.** Countable Zipper Fractal Interpolation Functions
  2.1 Construction of Countable Zipper Fractal Interpolation Function
  2.2 Stability Properties of Countable Zipper Fractal Interpolation Function
  2.3 A Parameterized Family of Zipper Fractal Functions and a Zipper Fractal Operator
  2.4 Extension of Zipper Fractal Operator
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  2.6 Conclusion

- **Chapter 3.** Countable Bivariate Fractal Interpolation Functions
  3.1 Construction of Countable Bivariate Fractal Interpolation Surfaces
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  3.3 A Parameterized Family of Bivariate Fractal Functions and Associated Fractal Operator
  3.4 Extension and Some Properties of Fractal Operator
  3.5 Conclusion
- **Chapter 4.** Multivariate Fractal Interpolation Function on Rectangular Grids
  4.1 A General Framework to Construct Multivariate Fractal Interpolation Functions
  4.2 A Parameterized Family of Multivariate Fractal Functions and Associated Fractal Operator
  4.3 On Approximation Aspects of Multivariate α-Fractal Functions
  4.4 Transfinite Multivariate α-Fractal Functions
  4.5 Dimensional Analysis of Multivariate α-Fractal Functions
  4.6 Fractional Integral of Continuous Multivariate α-Fractal Function
  4.7 Conclusion.
- **Chapter 5.** Smoothness Preserving Multivariate Fractal Interpolation Functions
  5.1 Smooth Multivariate α-Fractal Functions
  5.2 Multivariate Fractal Hermite Interpolation
  5.3 On a Constrained Multivariate Fractal Approximation
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- **Chapter 6.** Multivariate Fractal Functions in Lebesgue and Sobolev Spaces
  6.1 Multivariate α-Fractal Functions in Lebesgue Spaces
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- **Chapter 7.** Fractal Functions in Mixed Norm Spaces
  7.1 α-Fractal Functions in Mixed Norm Spaces
  7.2 Multivariate Kantorovich Operators in Mixed Lebesgue Spaces
  7.3 An Application of Bernstein-Kantorovich Operators in Fractal Approximation
  7.4 Conclusion

4. List of Publications

Published Articles

1. K.K. Pandey, P. Viswanathan, Multivariate Fractal Interpolation Functions: Some Approximation Aspects and an Associated Nonlinear Fractal Interpolation Operator, **Electronic Transactions on Numerical Analysis**, 55 (2022), 627-651, DOI: https://doi.org/10.1553/etna_vol55s627.

2. K.K. Pandey, P. Viswanathan, In Reference to a Self-Referential Approach Towards Smooth Multivariate Approximation, **Numerical Algorithms**, 91 (2022), 251–281, DOI: https://doi.org/10.1007/s11075-022-01261-7.

3. K.K. Pandey, P. Viswanathan, Multivariate Fractal Functions in Some Complete Function Spaces and Fractional Integral of Continuous Fractal Functions, **Fractal and Fractional** 5(4), 185 (2021), 19 pages, DOI: https://doi.org/10.3390/fractalfract5040185.

4. K.K. Pandey, P. Viswanathan, Countable Zipper Fractal Interpolation and Some Elementary Aspects of the Associated Nonlinear Zipper Fractal Operator, **Aequationes Mathematicae** 95 (2021), 175-200, DOI: https://doi.org/10.1007/s00010-020-00766-7.

Preprints

1. K.K. Pandey, N.A. Secelean, P. Viswanathan, On Bivariate Fractal Interpolation for Countable Data and Associated Nonlinear Fractal Operator (Review reports received from Demonstratio Mathematica, revised version under preparation). Arxiv: https://arxiv.org/abs/2010.05467.

2. K.K. Pandey, P. Viswanathan, Some Elementary Properties of Bernstein-Kantorovich Operators on Mixed Norm Lebesgue Spaces and Their Implications in Fractal Approximation (communicated to a journal for possible publication).

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